CONSTITUTIVE EQUATIONS OF STRUCTURALLY ORTHOTROPIC CYLINDRICAL SHELLS

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Successive, simplified equations of shells can be obtained by asymptotic splitting of the starting system [1-4]. They enable the solution of a wide class of practically important problems; however, for different approximate relations must be used for variabilities of the stressed state. As shown in this work, for structurally orthotropic cylindrical stringer and ring-framed shells this drawback can be eliminated by the method of constitutive equations [5]. Reference [6] is one of the first works in which a similar idea was used for isotropic shells.

1. In [2-4] it is shown that the equations of structurally orthotropic cylindrical shells can be further (compared with the isotropic case [1]) simplified. This is linked with the presence of an additional small parameter in them — the ratio of the flexural rigidities in different directions. Based on an analysis of real structures, in [2] the classes of stringer and ribbed shells, in particular, are singled out. For these shells the following relations are characteristic:

$$D_2/D_1 \equiv \varepsilon_2 \sim \varepsilon_1^2 \equiv D_2/(B_2R^2),\ D_3/D_1 \equiv \varepsilon_3 \sim \varepsilon_1,$$
$$B_2/B_1 \equiv \varepsilon_4 \sim 1,\ B_3/B_1 \equiv \varepsilon_5 \sim 1,\ e_1/R \equiv \varepsilon_6 \sim \varepsilon_1^2;$$
$$\varepsilon_2 \sim \varepsilon_1^{-1},\ \varepsilon_3 \sim 1,\ \varepsilon_4 \sim 1,\ \varepsilon_5 \sim 1,\ e_2/R \equiv \varepsilon_7 \sim \varepsilon_1^{1/2},$$

where B_i, $D_i (i = 1, 2)$ are the membrane and flexural rigidities; e_i is the distance from the center of mass of the collection of sections up to the center surface of the shell; the index 1 corresponds to longitudinal and the index 2 to annular supports; B_3, D_3 are the shear and torsional rigidities; and, R is the radius of the shell.

The main difference between the simplified equations obtained in [2-4] and the isotropic case consists of the fact that the equations of the theory of tapered shells also split into two equations, each of which is of fourth order with respect to the longitudinal coordinate.

We shall examine in detail the case of the static stressed state of a stringer shell [2]. For the characteristic of the variability of the potential function Φ along the coordinates we introduce the parameters α, β with the help of the relations

$$\Phi_\xi \sim \varepsilon_1^{-\alpha}\Phi,\ \Phi_\eta \sim \varepsilon_1^{-\beta}\Phi,$$

where $\xi = x/R$; $\eta = y/R$; and, x and y are the longitudinal and circular coordinates.

For small variability of the stressed—strained state (SSS) in the annular direction ($\beta = 0$), we obtain the following equations for the principal state ($\alpha = -0.5$)

$$\varepsilon_1^2\varepsilon_2\varepsilon_4\left(1 + \frac{\partial^2}{\partial\eta^2}\right)\Phi_{\eta\eta\eta\eta} + \Phi_{\xi\xi\xi\xi} = 0 \qquad (1.1)$$

and for the simple edge effect ($\alpha = 0.5$)

$$(\varepsilon_1^2 + \nu_{12}^2\varepsilon_6)\Phi_{\xi\xi\xi\xi} + \Phi - 2\nu_{12}\varepsilon_6\Phi_{\xi\xi} = 0, \qquad (1.2)$$

where ν is Poisson's ratio.

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For large variability of the SSS along the ring ($\beta = 0.5$) we have the equations

$$\alpha = 0: \left(\{1\} + \varepsilon_6 \frac{\partial^2}{\partial\eta^2}\right)^2 \Phi_{\xi\xi\xi\xi} + \varepsilon_1^2\varepsilon_4 \nabla_1^4 \Phi_{\eta\eta\eta\eta} = 0; \qquad (1.3)$$

$$\alpha = 0{,}5: A_2\Phi \equiv \nabla_2^4\Phi + \varepsilon_1^{-2}\left(\{1\} - \nu_{12}\varepsilon_6 \frac{\partial^2}{\partial\xi^2}\right)^2 \Phi + \varepsilon_6\left(\{2\} - \varepsilon_6 \frac{\partial^2}{\partial\eta^2} - 2\nu_{12}\frac{\partial^2}{\partial\xi^2}\right)\Phi_{\eta\eta} = 0, \qquad (1.4)$$

where

$$\nabla_1 = \frac{\partial^4}{\partial\xi^4} + 2\varepsilon_3 \frac{\partial^4}{\partial\xi^2\partial\eta^2} + \varepsilon_2 \frac{\partial^4}{\partial\eta^4}; \quad \nabla_2 = \frac{\partial^4}{\partial\xi^4} + 2\varepsilon_4\varepsilon_5^{-1} \frac{\partial^4}{\partial\xi^2\partial\eta^2} + \varepsilon_4 \frac{\partial^4}{\partial\eta^4}.$$

Finally, when $\beta > 0.5$ the starting equation splits into equations for bending and plane deformation of a stringer plate, which are obtained from (1.3) and (1.4), if the terms in the braces are dropped in them.

We shall now try to construct the constitutive equations, suitable for any variability of the SSR. It is not difficult to see that (1.4) already satisfies this condition, since it includes the equation of the edge effect (1.2) and of the plane SSS of the plate. To transform the equation (1.3) into a constitutive equation, the operator ∇_1 must be replaced by the following operator:

$$\nabla_{10} = \frac{\partial^4}{\partial\xi^4} + 2\varepsilon_3 \frac{\partial^4}{\partial\xi^2\partial\eta^2} + \varepsilon_2\left(1 + \frac{\partial^2}{\partial\eta^2}\right)^2.$$

Then the equation

$$A_1\Phi \equiv \varepsilon_4\varepsilon_1^2\nabla_{10}^4\Phi_{\eta\eta\eta\eta} + \left(1 + \varepsilon_6\frac{\partial^2}{\partial\eta^2}\right)^2\Phi_{\xi\xi\xi\xi} = 0 \qquad (1.5)$$

includes the equation of the principal state (1.1) and bending of the plate.

The principal, from the energy standpoint, deformations for SSS described by (1.3) are the curvatures κ_1, κ_2 and the torsion κ_{12}, and also the deformation in the longitudinal direction ε_{11}. For (1.4) the principal deformations are κ_1, ε_{11}, ε_{22}, ε_{12}.

Using the limiting equation [2] for a ring-framed shell we have the following constitutive equations:

$$A_3\Phi = \frac{\partial^4}{\partial\eta^4}\left[\left(\varepsilon_1^2\varepsilon_2 + \varepsilon_7^2\right)\frac{\partial^4}{\partial\xi^4} + 2\left(\varepsilon_1^2\varepsilon_2\varepsilon_5 - \nu_{12}\varepsilon_7\right)\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \left(\varepsilon_1^2\varepsilon_2\varepsilon_4 + \nu_{12}^2\varepsilon_7^2\right)\left(\frac{\partial^2}{\partial\eta^2} + \{1\}\right)^2\right] + \varepsilon_7\frac{\partial^4}{\partial\xi^2\partial\eta^2}\left(\frac{\partial^2}{\partial\xi^2} - \nu_{12}\frac{\partial^2}{\partial\eta^2} - 2\nu_{12}\right)\Phi + \frac{\partial^4\Phi}{\partial\xi^4} = 0; \qquad (1.6)$$

$$A_4\Phi \equiv \varepsilon_1^2\nabla_{10}^4\Phi + \left(1 + \varepsilon_7\frac{\partial^2}{\partial\eta^2}\right)^2\Phi = 0. \qquad (1.7)$$

Equation (1.7) is the same as the limiting equation presented in [2], and in (1.6) the term in the braces has been added. The relation (1.6) includes the equations of the principal state, the stressed state with a large index of variability in the annular direction and predominantly tangential deformation of the plate, and the relation (1.7) describes the edge effect, the stress state with a large index of variability in the longitudinal direction and predominantly flexural deformation of the plate. The principal deformations for SSS, described by (1.6) and (1.7), are ε_{11}, ε_{22}, ε_{12}, κ_2 and ε_{22}, κ_1, κ_2, κ_{12}, respectively.

2. The constitutive equations can also be constructed for the problems of stability and dynamics, and the limiting equations for structurally orthotropic stringer and ring-framed shells are presented in [3, 4].

The constitutive equations of stability for a stringer shell can be represented in the form

$$A_1\Phi + \varepsilon_4\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial^2}{\partial\eta^2} + \{1\}\right)\left(\overline{T}_{10}\frac{\partial^2}{\partial\xi^2} + 2\overline{S}_0\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \overline{T}_{20}\frac{\partial^2}{\partial\eta^2}\right)\Phi = 0; \qquad (2.1)$$

TABLE 1

Conditions for (1.6) or (2.3) or (2.7)	General conditions	Conditions for (1.7) or (2.4) or (2.8)	Conditions for (1.6) or (2.3) or (2.7)	General conditions	Conditions for (1.7) or (2.4) or (2.8)
u, v	w	w_ξ	u	S, w	w_ξ
T_1, v	w	w_ξ	T_1	S, w	w_ξ
u, v	w	M_1	u	S, w	M_1
T_1, v	w	M_1	T_1	S, w	M_1

$$A_2\Phi + \{\varepsilon_1^{-2}\overline{T}_{10}\Phi\} = 0. \tag{2.2}$$

For ring-framed shells we have

$$A_3\Phi + \overline{T}_{20}\frac{\partial^2}{\partial\eta^2}\left[\frac{\partial^4}{\partial\xi^4} + 2\varepsilon_3\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \varepsilon_2\left(1 + \frac{\partial^2}{\partial\eta^2}\right)^2\right]\Phi + \left\{\left(\frac{\partial^2}{\partial\eta^2} + 1\right)\left(2\overline{S}_0\frac{\partial^2}{\partial\xi\partial\eta} + \overline{T}_{10}\frac{\partial^2}{\partial\xi^2}\right)\Phi\right\} = 0; \tag{2.3}$$

$$A_4\Phi + \left(\overline{T}_{10}\frac{\partial^2}{\partial\xi^2} + 2\overline{S}_0\frac{\partial^2}{\partial\xi\partial\eta} + \overline{T}_{20}\frac{\partial^2}{\partial\xi^2}\right)\Phi = 0, \tag{2.4}$$

where $\{\overline{T}_{10};\ \overline{S}_0;\ \overline{T}_{20}\} = \{T_{10};\ S_0;\ T_{20}\}/B_2;\ T_{10},\ T_{20},\ S_0$ are the axial, annular, and shear forces in the subcritical state.

Equations (2.1)-(2.3) differ from those presented in [3] by the terms in the braces, and Eq. (2.4) is the same as the equation presented in [3].

In studying the oscillations of stringer shells the constitutive equations, which hold for any frequencies and variabilities, assume the form

$$A_1\Phi - \{\omega^2 A_{11}\Phi\} \equiv A_1\Phi - \left\{\omega^2\left[\left[\frac{\partial^4}{\partial\xi^4} + 2\varepsilon_5^{-1}\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \varepsilon_4\left(\frac{\partial^2}{\partial\eta^2} + 1\right)\frac{\partial^2}{\partial\eta^2}\right] + \varepsilon_1^2\varepsilon_4\left[(a_0 + 1)\frac{\partial^2}{\partial\xi^2} + (a_0 + \varepsilon_4)\frac{\partial^2}{\partial\eta^2}\right]\omega^2 + \varepsilon_4 a_0\omega^4\right]\right\}\Phi = 0; \tag{2.5}$$

$$\varepsilon_1^2 A_2\Phi - \omega^2\Phi = 0. \tag{2.6}$$

Here ω is the frequency of the oscillations, and $a_0 = 2(\varepsilon_5^{-1} + \nu_{12})/(1 - \nu_{12}\nu_{21})$. For ring-framed shells we have

$$A_3\Phi - \{\omega^2 A_{11}\Phi\} = 0; \tag{2.7}$$

$$A_4\Phi - \omega^2\Phi = 0. \tag{2.8}$$

In [4] equations which follow from (2.5)-(2.8), if the terms in braces are dropped in them, are presented for the lower part of the spectrum; for high-frequency oscillations $A_{11}\Phi = 0$.

3. In the formulation of the boundary-value problems for the constitutive equations constructed, the synthesis of separated boundary conditions is also used. It turns out that the boundary conditions for the constitutive equations of stringer shells cannot be separated. For ring-framed shells such separation is possible. Let us examine, for example, the following variant of the boundary conditions:

$$\text{for } \xi = 0, l \quad u = \tilde{u},\ w = \tilde{w},\ S = \tilde{S},\ M_1 = \tilde{M},$$

where $\ell = L/R$; ℓ is the length of the shell; u and w are longitudinal and normal displacements; S is the shear force; M_1 is the longitudinal bending moment; $\tilde{u}$, $\tilde{w}$, $\tilde{S}$, $\tilde{M}$ are the fixed edge values of the corresponding components of the SSSR.

Let us examine (1.6) and (1.7). For the principal state, conditions on u and S must be given [1, 7]; for the equations with the predominant variability in the annular direction conditions on u and w must be given [7]; and, for a plane SSS conditions on u and S must be given. Therefore, for the constitutive equation (1.6), including the indicated

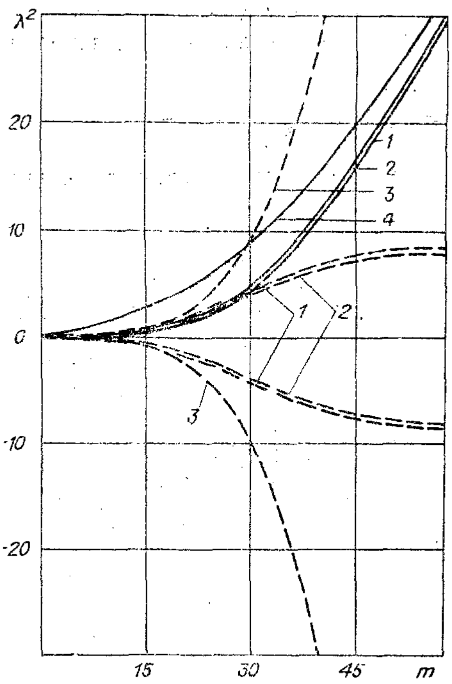

Fig. 1

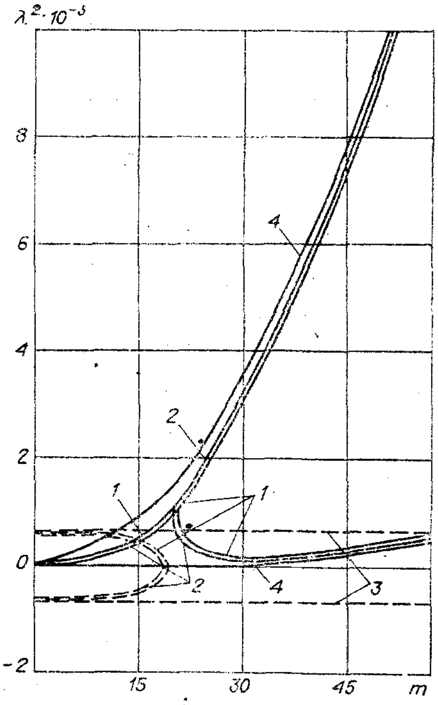

Fig. 2

relations, boundary conditions on u, S, and w must be given. Analogously, we find that for (1.7) it is necessary to state boundary conditions for S, w, and w_ξ. Finally, we find that the conditions on S and w must be the same for both equations, the conditions on u revert to (1.6), while the condition on w_ξ reverts to (1.7).

The final results are presented in Table 1, where v is the ring displacement, and T_1 is the longitudinal force.

The procedure described in [8] can be used to solve specific problems. Namely, after the solutions of the boundary layer type are constructed they must be eliminated from the boundary conditions. As a result there remain the boundary conditions for equations of the principal-state type.

4. To determine the region of applicability of the constitutive equations, we compared the squares of the roots of the characteristic equations corresponding to the equations of a stringer shell (1.4) and (1.5) with the exact values found numerically. The potential function Φ is represented in the form

$$\Phi = C \exp(\lambda \xi) \cos(m\eta).$$

The following values of the geometric-rigidity parameters were used:

$$\varepsilon_1 = 2{,}2\cdot 10^{-6},\ \varepsilon_2 = 10^{-4},\ \varepsilon_3 = 10^{-2},\ \varepsilon_4 = 0{,}6,$$
$$\varepsilon_5 = 0{,}3, \qquad \varepsilon_6 = \varepsilon_7 = 0,\ \nu_{12} = 0{,}2. \qquad (4.1)$$

The results of the comparison are presented in Figs. 1 and 2, where the solid lines show the real parts of λ^2 and the broken lines show the imarginary parts. The numbers 1 denote the exact solution; the numbers 2-4 denote the solutions based on (1.4) (Fig. 1), (1.5) (Fig. 2), (1.1) (Fig. 2), and (1.2) (Fig. 2); Fig. 1 is for bending of the plate and Fig. 2 is for plane SSS of the plate.

It is evident that the constitutive equations are indeed applicable for all m and realize smooth joining of the solutions valid for small and large variabilities of the SSS.

We also carried out calculations using the parameter (4.1) (characteristic for a real structure), when ε_1 varied from 10^{-6} to 10^{-5}, while ε_2 varied from 10^{-4} to 10^{-3}. The remaining parameters, as shown in [2-4], do not significantly affect the nature of the asymptotic expansions. The error in the constitutive equations in all cases did not exceed 5% in the entire range of variation of m.

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